

Chaotic Traveling Waves in a Coupled Map Lattice

Kunihiko KANEKO *

Department of Pure and Applied Sciences,

College of Arts and Sciences

University of Tokyo, Komaba, Meguro-ku, Tokyo 153, JAPAN

Abstract

Traveling waves triggered by a phase slip in coupled map lattices are studied. A local phase slip affects globally the system, which is in strong contrast with kink propagation. Attractors with different velocities coexist, and form quantized bands determined by the number of phase slips. The mechanism and statistical and dynamical characters are studied with the use of spatial asymmetry, basin volume ratio, Lyapunov spectra, and mutual information. If the system size is not far from an integer multiple of the selected wavelength, attractors are tori, while weak chaos remains otherwise, which induces chaotic modulation of waves or a chaotic itinerancy of traveling states. In the itinerancy, the residence time distribution obeys the power law distribution, implying the existence of a long-ranged correlation. Supertransients before the formation of traveling waves are noted in the high nonlinearity regime. In the weaker nonlinearity regime corresponding to the frozen random pattern, we have found fluctuation of domain sizes and Brownian-like motion of domains. Propagation of chaotic domains by phase slips is also found. Relevance of our discoveries to Bénard convection experiments and possible applications to information processing are briefly discussed.

1 Introduction

Spatiotemporal chaos (STC) is high-dimensional chaos which involves spatial pattern dynamics. It covers turbulent phenomena in general, including Bénard convection, electric convection in liquid crystals, boiling, combustion, magnetohydrodynamic turbulence in plasmas, solid-state physics (Josephson junction arrays, spin wave turbulence, charge density waves and so on), optics, chemical reactions with spatial structures, etc. It is also important in biological information processing with nonlinear elements like neural dynamics.

The coupled map lattice (CML) is a dynamical system with discrete time (“map”), discrete space (“lattice”), and a continuous state. It usually consists of dynamical elements on a lattice interacting (“coupled”) among suitably chosen sets of other elements [1-15]. The CML was originally proposed as a simple model for spatiotemporal chaos.

Modelling through a CML is carried out as follows: Choose essential procedures which are essential for the spatially extended dynamics, and then replace each procedure by a parallel dynamics on a lattice. The CML dynamics is obtained by successive application of each procedure. As an example, assume that you have a phenomenon, created by a local chaotic process and diffusion. Examples can be seen in convection, chemical turbulence, and so on. In CML approach, we reduce the phenomena into local chaos and diffusion

*E-mail address : chaos@tansei.cc.u-tokyo.ac.jp

processes. If we choose a logistic map $x'_n(i) = f(x_n(i))$ ($f(y) = 1 - ay^2$) to represent chaos, and a discrete Laplacian operator for the diffusion, our CML is given by

$$x_{n+1}(i) = (1 - \epsilon)f(x_n(i)) + \epsilon/2(f(x_n(i+1)) + f(x_n(i-1))) \quad (1)$$

One of the merits of the CML approach lies in its predictive power of novel qualitative universality classes, without being bothered by the details of phenomenology. Classes discovered thus far include spatial bifurcation, frozen random chaos, pattern selection with suppression of chaos, spatiotemporal intermittency, soliton turbulence, quasistationary supertransients, and so on [2-7].

In the present paper we report a novel universality class in CML, which is related to recent experiments in fluid convection and liquid crystals: (chaotic) traveling waves. We study the qualitative and quantitative nature of the chaotic traveling wave, with the help of the Lyapunov analysis and co-moving mutual information flow.

In our model (1), observed domain structures are temporally frozen when the coupling ϵ is small ($< .45$), as has been studied in detail in [5]. For larger couplings, domain structures are no longer fixed in space, but can move with some velocity. For weak nonlinearity ($a < a_{ps} \approx 1.55$), the motion of a domain is rather irregular and Brownian-like, while pattern selection yielding regular waves is found for larger values of the nonlinearity. These two regions correspond to the frozen random phase and (frozen) pattern selection in the weaker coupling regime [5], respectively. Our novel discovery here is that the pattern is no longer frozen but can slowly move.

The organization of the paper is as follows. In section 2, the coexistence of traveling-wave attractors with different velocities is shown. The quantization of selected velocities is noted, and the basin volume for each attractor is investigated. The mechanism of traveling is attributed to the existence of phase slips, phase-advancing units, as will be studied in detail in section 3. The traveling wave suppresses chaos almost completely, as is confirmed by the Lyapunov analysis in section 4. When the size is not close to an integer multiple of the selected wavelength, weak chaos remains, which induce the modulation of traveling wave or a chaotic itinerancy over different traveling wave states due to chaotic frustration in the pattern. The long-term correlation of the itinerancy is studied in section 5. The flow of information in the traveling wave is characterized by co-moving mutual information flow in section 6. Quasistationary supertransients before falling on a traveling wave attractor are studied in section 7. Switching among attractors by a local input is studied in section 8, where it is shown that a single input can induce a transition of an attractor's velocity and thus affect the entire lattice. In a weak nonlinearity regime, the motion of a domain is no longer regular. The Brownian-like motion of domains is studied in section 9. If the local dynamics is not chaotic, but periodic with the period 2^n , we can have traveling kinks in the strong coupling regime. These kinks are localized in space and do not have a global influence, in contrast with the phase slips, as will be shown in section 10. Discussions and a summary are given in section 11 [1].

2 Selection of discrete velocities

In the CML (1), only a few patterns with some wavenumbers are selected for large nonlinearity ($a > 1.55$ for $\epsilon = .5$). Examples of attractors are given in Fig.1. Besides the non-traveling pattern, there are moving patterns which form a traveling wave. We note that such traveling attractors are not observed in the weak coupling regime ($\epsilon < .4$). The selected velocities of the attractors in the examples are rather low, in the order of 10^{-3} .

As can be seen in Fig.1, attractors with different velocities of waves coexist. In the simulation, the admissible velocities v_p for the attractors lie in narrow bands located at $+v_1, +v_2, \dots, +v_k$ (e.g., $8v_1 < v_p < 1.2v_1$). For example, $v_1 = .95 \times 10^{-4}$, $v_2 = 1.9 \times 10^{-3}$.

and $v_3 = 2.9 \times 10^{-3}$, $v_4 = 3.9 \times 10^{-3}$, for $a = 1.72$, $\epsilon = .5$, and $N = 100$. No attractors exist with $v_p \approx 0$ but $v_p \neq 0$. There is a clear gap between the velocities of the attractors in each velocity band; No attractors are found with different velocity from these bands around v_k . For all parameters, v_k is approximately proportional to k .

—Fig.1—

One might argue that this discreteness of the velocity bands may be an artifact of our model, which is discrete in both space and time. Since the speed is very slow (i.e., the order of 10^{-3} site per step), it is not easy to imagine a mechanism to which our original discreteness (the order of 1 site per step) is relevant. To examine possible effects of the spatial discreteness, we have also simulated a CML with a much longer coupling range, following the method of §7.5 in [5]; i.e., a repetition of the diffusion procedure in the CML of I_D times per local nonlinear mapping procedure. With the increase of I_D , the range of the diffusion is increased, making our attractor spatially much smoother, approaching a continuous space limit. Our traveling attractors have then, much longer wavelengths, and have higher quantized speeds. For example the speed band at v_1 is amplified roughly 4 times by choosing $I_D = 8$ (for $a = 1.72, \epsilon = .5, N = 200$). Thus the discreteness in space is not relevant to the discrete selection of velocities.

The wavelength of a pattern is almost independent of the velocity of an attractor. The velocity is governed not by the (spatial) frequency but by the form of the wave. Since our model has mirror symmetry, a traveling wave attractor must break the spatial symmetry. The wave form is spatially asymmetric. Here this spatial (a)symmetry is not a local property. Indeed, the waveform differs by domains of unit wavelength. The asymmetry is defined only through the average over the total lattice. We have measured the spatial asymmetry by

$$s \equiv \langle \frac{1}{N} \sum_{j=1}^N (x_n(j+1) - x_n(j))^3 \rangle \quad (2)$$

with the long time average $\langle \dots \rangle$. This third power is chosen just because it is the simplest moment of an odd power, since the first power $\sum_{j=1}^N (x_n(j+1) - x_n(j))$ vanishes due to the periodic boundary conditions. In the present paper the velocity of an attractor is estimated by virtue of the following algorithm: Find the minimum k such that $\sum_{j=1}^N (x_{n+2m}(j) - x_n(j-k))^2$ is a minimum. Up to some value of $2m$, the minimum is found for the lattice displacement $k = 0$. If the attractor is moving, at a certain delay $2m$, the minimum is not obtained for $k = 0$, but for $k = \pm 1$. With the help of this delay the speed of the pattern is estimated as $\pm 1/(2m)$. In Fig.2, we have measured the above $2m$ over time 160000 steps, after discarding 10000 initial transients, to obtain an accurate for the average velocity.

The relationship between s and v_p is shown in Fig.2, for several values of the parameter a . If $a \gtrsim 1.74 \approx a_{tr}^1$, the relationship is rather simple. The velocity of an attractor turns out to be proportional to its asymmetry s , as is plotted in Fig.2c)d). Here we note that there is a gap of velocity between frozen attractors ($v_p = 0$) and traveling attractors. For an attractor with velocity $v = 0$, s is zero within numerical accuracy. Thus spatial symmetry is attained through the attraction to the non-traveling attractor, starting from an initial condition with spatial asymmetry. Again, this spatial symmetry is not a local but a global property. Indeed, for each domain over a single wavelength, its waveform is not generally mirror symmetric. The asymmetry in each wave form is cancelled through the summation over the entire lattice. This attainment of self-organized symmetry is possible under the existence of traveling attractors. Indeed, for a weaker coupling regime without a traveling attractor, all attractors have a fixed structure [5], but they are not generally

spatially symmetric. Spatial asymmetry in the initial conditions is not eliminated in this case.

For $a < 1.74$, the relationship between s and v_p is more complicated. Attractors with $v_p = 0$ can have a small non-vanishing asymmetry. The self-organized symmetry is not complete. The velocity gap between frozen attractors and moving ones is not seen. Furthermore the linear relationship between v_p and s does not hold, although we can see a band structure of velocities. One of the reasons of this complication is coexistence of attractors with different periods (or frequencies), as will be studied in the next section.

—Fig. 2 a)b)c)d)—

For random initial conditions, the probability to hit an attractor with the velocity 0 or $\pm v_1$ is rather high. In Fig.3, we have measured the basin volume ratio for attractors of different velocities. A band structure of admissible velocities is found. In each band there are discrete sets of admissible velocities. We have confirmed that there are many attractors with different velocities within each band by running a long-time simulation.

—Fig. 3 a)b)—

As the velocity of an attractor increases, its basin volume shrinks rather drastically (see Table I). The basin volume for v_1 is often rather large. As is shown in Fig.3, basin volumes decrease (approximately) in a Gaussian form with the velocity of the band ($\exp(-K^2 \times \text{const.})$ for attractors in the band $v_p = Kv_1$). This Gaussian decrease is generally observed for any parameter value, although the basin ratios for the fixed and v_1 attractors may vary.

Table I: Velocity, asymmetry, and basin volume of fixed and traveling attractors. $a = 1.73$, and $\epsilon = .5$. 500 attractors from random initial conditions are chosen to estimate the basin volume ratio.

velocity	0	$\pm v_1 = .95$	$\pm v_2 = 1.95$	$\pm v_3 = 2.9$	$\pm v_4 = 3.8$	$\times 10^{-3}$
asymmetry s	0	2.5	5.3	8.3	12	$\times 10^{-5}$
basin volume ratio	34.6%	23.7%	7.2%	1.6%	0.2%	

Dependences of the velocity v_p on the parameter a and size N are given in Figs.4-5. In these figures, we have measured the velocity by taking the average over 160000 time steps, after discarding 800000 transients starting from several initial conditions. These averaged velocities are plotted for $1.6 < a < 1.85$ for $N = 50$ (in Fig.4), while they are plotted over $10 < N < 250$ for $a = 1.73$, in Fig.5. We can see the selection of discrete velocities rather well. Velocities lie in a narrow band around v_k .

—Fig. 4 —

—Fig. 5 —

As is given in Fig.5, selected velocities slowly decrease with the system size. Our system has a selected wavelength R , and the fractional part of $\frac{N}{R}$ is rather essential for the nature of traveling wave². Except for this additional dependence, the velocity decrease is roughly fitted by $1/\sqrt{N}$ up to $N = 200$. We also note that higher bands successively

¹See section 7 for a_{tr} , where possible mechanism for the change of s - v relationship at a_{tr} is discussed.

appear (v_k with larger k), with the increase of the system size, although the basin volume for such higher bands is rather small due to the Gaussian decay as shown in Fig.3.

As has been reported, no traveling state has been observed for $\epsilon < \epsilon_c \approx .402$. We note that the velocity does not go to zero as ϵ approaches ϵ_c from above. For $.402 < \epsilon < .45$ the velocity lies between 1.0×10^{-3} and 1.8×10^{-3} without displaying any symptoms of a decrease. Rather, the basin volume for the traveling attractor vanishes with $\epsilon \rightarrow \epsilon_c$, which is the reason why only non-traveling attractors are observed for $\epsilon < \epsilon_c$. The basin volume for traveling states (i.e., all attractors with non-zero velocities) is shown in Fig. 6.

— Fig. 6 —

3 Phase Slips: Local units for global traveling wave

To understand the mechanism of this velocity selection, we note that $x_n(i)$ oscillates in time. For $\epsilon = .5$, the oscillation is almost periodic and the period is very close to 4. Then one can assign a phase of oscillation to a lattice site i relative to $(x_n(i), x_n(i+1))$. It is possible to assign a phase change $m'\pi/2$ ($m' = \pm 1$) between the lattice site i and the lattice site $i+j$ in a neighboring domain, according to the order of the period-4 like motion.

—Fig.7 —

When there is a phase gap of 2π between sites i and $i+\ell$, it is numerically found that this interval unit $[i, i+\ell]$ maintains the traveling wave. For example, in the attractor with velocity v_1 in Fig.1, the oscillation is close to period 4, with slow quasiperiodic modulation. For periodic boundary conditions, the total phase change should be $2k\pi$. The velocity is zero for an attractor with $k = 0$. If $k = 1$, there must be a sequence of 5 domains with phases $0, \pi/2, \pi, 3\pi/2, 2\pi$ for corresponding lattice sites i (Fig.7). This unit is a phase slip of 2π . A phase slip with a negative sign is defined by the mirror-symmetric pattern of a positive one. An attractor with the velocity of the band v_k has exactly k (positive) phase slips, in other words, $2k\pi$ phase change over the total lattice. (k equals the number of positive phase slips subtracted by negative ones). Among attractors with the same number of phase slips, there can be various configurations of domains. The velocity variance among attractors within the same band depends on this configuration. Since a phase slip is localized in space, one might think that the movement is a local phenomenon like soliton propagation. This is not the case. In the present case, this phase slip must pull all the other regions to make them travel, changing the phases of oscillations of all lattice points. Thus a local slip influences globally all lattice points. Our dynamics gives a connection between local and global dynamics. One clear manifestation of the global aspect is the additivity of velocity. In our system, the velocity of the wave is proportional to the number of phase slips. This proportionality gives a clear distinction between our dynamics and soliton-type dynamics, where, of course, the velocity of a soliton does not increase with the number of solitons present.

The phase of oscillation can clearly be seen with the use of spatial return maps, 2-dimensional plots of $(x_n(i), x_n(i+1))$. When there is a phase slip, the spatial return map shows a curve as in Fig.8. A point $(x_n(i), x_n(i+1))$ rotates clockwise with time when there is a phase slip, while the point does not rotate for a non-traveling attractor. When there are two phase slips, the rotation speed is twice in addition to a slight change of the

²See section 4 for a novel dynamical state which appears when there is a mismatch between the size and the wavelength.

curve. We note that the motion is smooth without any remarkable change of rotation speed even when the lattice site lies at the phase slip region. In Fig.8., we have plotted spatial return maps for attractors with 1,2,3, and 4 phase slips. In the figure the system size is 64 lattice sites while each phase slip requires 16 lattice sites. Thus the attractor with 4 slips (see fig. 8d) consists only of a sequence of 4 repeated phase slip patterns. For attractors with less than 4 slips, there can be variable configurations of domains other than the phase slips. Depending on the configuration, the spatiotemporal return maps are different.

When the return map shows a closed curve, the attractor is on a projection of a 2-dimensional torus. This is the case in the state consisting only of phase slips (see fig.8d). In general, the curve is not closed and the return map forms surface rather than a curve (see fig. 8.a-c), suggesting a higher-dimensional attractor. Indeed, in Fig 8c), for example, we can see clearly another frequency modulation. As will be confirmed in §4, the attractor is on a higher-dimensional torus. Frequencies of quasiperiodic modulation depend on the number of phase slips and the configuration of the domains.

—Fig. 8 —

Some of the numerical results in the previous section are explained by the phase slip mechanism in this section.

The proportionality between the asymmetry and the velocity can partially be explained by the fact that each (positive) phase slip gives rise to a certain contribution to the asymmetry s . In Fig.2c)d), however, the proportionality holds even in a level within each band where the number of slips is identical. The asymmetry can depend on the configurations of domains, besides the number of slips. So far it is not clear why the proportionality holds even for such small changes of the asymmetry by the configurations, when the nonlinearity is large.

The basin volume vs. the velocity: Let us assume that by a random initial condition, a phase change between two domains ($\pm\pi/2$) are randomly assigned. (We have to impose the constraint that its sum should be a multiple of integers of 2π , but this is not important for the following rough estimate). Then the probability for the sum of the phase change obeys the binomial distribution. For large N , the probability to have K phase slips is estimated as $\exp(-(K/\sigma)^2)$ with $\sigma \propto \sqrt{N}$. Thus the probability to have K phase slips is expected to decay with a Gaussian form with K . Thus the Gaussian form of the basin volume (see Fig.3b) and the $1/\sqrt{N}$ dependence of the velocity (in Fig. 4) are explained.

4 Chaos and Quasiperiodicity in the traveling wave

To examine the dynamics of our attractors, Lyapunov spectra are measured numerically through the product of Jacobi matrices [4], and are plotted in Fig. 9. For most parameters ($1.65 < a < 2.0$) and sizes, the maximal exponent is zero, irrespective of the velocity of attractors. Thus chaos is completely eliminated by pattern selection, and the attractor is a torus. As is expected from the spatial return maps, the attractor can be a higher-dimensional torus with more than one null exponent. Between attractors with $v = 0$ and $v = v_p$, there are only slight differences in Lyapunov spectra (see Fig. 9).

—Fig. 9 —

In our model a (traveling) pattern is selected such that it eliminates chaos (almost) completely. If chaos were not sufficiently eliminated, it would be impossible to sustain a spatially periodic pattern during the course of propagation. Such elimination of chaos

is not possible for every wave pattern, since our dynamics has topological chaos. In our system a wavelength R is selected. When the size N is not close to a multiple of the selected wavelength R , there can remain some frustration in any pattern configuration, and weak chaos can be observed.

In narrow parameter regimes, we have seen a chaotic traveling waves for some sizes. For example, very weak chaos is observed around $a \approx 1.70$, if N is large, as is shown in Fig. 10. The corresponding spatial return map (see Fig.11) consists of curves (corresponding to regular traveling) and scattered points (corresponding to chaotic modulation). It should be noted that the chaotic modulation propagates in the opposite direction as the traveling wave.

Lyapunov spectra are given in Fig.12, where few positive exponents exist in the traveling wave attractor. The number of positive exponents is small ($1 \sim 3$) compared with the system size N . Chaos, localized in a domain, propagates as a modulation of the wave, as is shown in Fig.10. We also note that the spectrum is almost flat near $\lambda \approx 0$. The propagating wave leads to a Goldstone mode giving rise to a null exponent.

—Fig.10 —

—Fig.11 —

—Fig.12 —

5 Chaotic Itinerancy of Traveling Waves

As is shown in the previous section, there remains some frustration when forming a wave pattern if the ratio N/R is far from an integer. When the frustration due to this mismatch between the size and wavelength is large, it leads to spontaneous switching among patterns (see Fig. 13). This spontaneous switching arises from chaotic motion of each pattern, and may be regarded as a novel class of chaotic itinerancy [13]. Global interaction is believed to be necessary to obtain a chaotic itinerancy [13]. Although the interaction of our model is local, the phase slip globally influences all the lattice points, and thus satisfies the condition for chaotic itinerancy.

Only few remnants of curves (corresponding to the traveling structure) can be seen in the spatial return map (see Fig.14), while scattered parts are more dominant than in the chaotic traveling in the previous section. The direction of rotation also changes with time, through the scattered points. Both amplitudes and phases of oscillations are modulated strongly here.

—Fig.13 —

—Fig.14 —

For the spontaneous switching, we need some kind of modulation of the wave. Indeed, each waveform starts to be rather irregular in space and time in advance to the switching. The wavelength, on the other hand, is not affected by the course of this switching process. In general, there can be three types of modulation of the wave; frequency, phase, and amplitude modulations. In our example, frequency is hardly modulated (as is seen in the invariance of wavelength through the switching), while the phase modulation (following the amplitude one) is essential to the spontaneous switch of traveling states.

The switching occurs through the creation or destruction of a phase slip. Frustration in a pattern leads to the distortion of a phase slip, inducing chaotic motion. This chaotic

motion breaks the phase slip. On the other hand, there can be the creation of a slip by chaotic modulation of the phase of oscillation. This creation or destruction of a phase slip is a local process, but influences globally the velocity of the traveling wave.

In chaotic itinerancy, long time residence at a quasi-stable state is often noted. We have measured the residence time distribution of a state with a given velocity. As is shown in Fig.15, all the residence time distributions $P_k(t)$ of a k -phase-slip state (for $k = 0, \pm 1, \pm 2$; i.e., $v_p = 0, v_p = \pm v_1, v_p = \pm 2v_1$) obey the power law $P_k(t) \approx t^{-\alpha}$ with $\alpha \approx 1$. This power-law dependence clearly indicates the long time residence at each traveling state. Similar power-law dependence of a quasistable state has already been found for spatiotemporal intermittency in a CML [5], although the power itself is clearly distinct.

Lyapunov spectra for this frustration-induced chaos are shown in Fig.16. The number of positive exponents is again very few (3 in the figure), whose magnitudes are very small. The chaos by the frustration is very weak and low-dimensional. The spectra have a plateau at the null exponent, implying the existence of a Goldstone mode by traveling wave. As seen in the previous section the accumulation at null exponent is characteristic of a (chaotic) system with a traveling wave.

For larger system sizes, chaotic itinerancy of waves is hardly observed. The system settles down to a frozen or traveling pattern after transients. Since the number of chaotic modes is few ($O(1)$), the frustration per degree of freedom is thought to decrease with N . The distortion due to the mismatch of phases is still there, but it is distributed over a large size and is too weak to switch the pattern. The remnant frustration in a traveling wave leads to chaotic modulation of wave as is studied in §4.

—Fig. 15 —

—Fig. 16 —

6 Co-Moving Mutual Information Flow

Co-moving mutual information is often useful for measuring correlations in space and time [4]. From the joint probability $P(x_n(i), x_{n+m}(i+j))$, we have calculated the mutual information

$$I(m, j) = \int dx_n(i) dx_{n+m}(i+j) P(x_n(i), x_{n+m}(i+j)) \log \frac{P(x_n(i), x_{n+m}(i+j))}{P(x_n(i))P(x_{n+m}(i+j))}. \quad (3)$$

In a traveling wave, we have peaks in $I(m, j)$ at $I(t, v_p t)$ for an attractor traveling with v_p . For a quasiperiodic attractor the peak height does not decay with the time delay t , while it slowly decays for a chaotic attractor. The transmission of correlations can clearly be seen.

In a chaotic attractor, however, there is also propagation of small modulations on the traveling wave. As can be seen in Fig.10, this propagation is in the opposite direction to the wave. From the above mutual information, this reverse propagation could not easily be measured so far. The propagation of chaotic modulation implies the flow of information created by chaos [16]. One way to measure this information creation may be the use of three point mutual information with the use of $P(x_n(i), x_{n+m}(i+j), x_{n+m+\ell}(i+j))$ [17], while another possible way of charactering a chaotic traveling wave is the use of the co-moving Lyapunov exponent [4]. A slight increase of the exponent at the traveling velocity is observed. In our case, however, chaos is too weak to give a quantitative distinction.

—Fig. 17 —

The mutual information in the chaotic itinerancy decays with time and space, without any peaks at some velocity. By the switching process, all local traveling structures are smeared out, leading to the destruction of peaks in the mutual information at some velocities (see Fig.18).

—Fig. 18 —

7 Chaotic Transients before the formation of Traveling Waves

To fall on a traveling (or fixed) attractor, the velocities of all local domains of unit wavelength must coincide. Thus it is expected that the transient time before falling on an attractor may increase with the system size. As for the transient behavior, our transient wave phase splits into the following two regimes.

(i) For medium nonlinearity regime ($a < a_{tr} \approx 1.74$), the transient length increases at most with the power of N . Indeed, local traveling wave patterns are formed within a few time steps. Before hitting the final attractor, these local waves are slightly modulated to form a global consistency. The formation of a global wave structure occurs for time steps smaller than $O(N)$. We need time steps in the order of $O(N)$ for the slight modulation to adjust the phases of all domains. (see Fig. 19a)b) for spacetime diagram).

(ii) For larger nonlinearity ($a > a_{tr}$), there are long-lived chaotic transients before our system falls on a traveling-wave attractor. The transient length increases with the system size rather rapidly: the increase is roughly estimated by $\exp(const. \times N)$ [14], although some (number-theoretically) irregular variation remains. In the transient process, the dynamics is strongly chaotic, and is attributed to "fully developed spatiotemporal chaos" in [5]³. Lyapunov spectra during the transients are shown in Fig.20, in contrast with the spectra of an attractor. This transient process is quasistationary (see Fig. 19c for spacetime diagram); No gradual decay is observed for dynamical quantifiers such as the short-time Lyapunov exponent [11] or Kolmogorov-Sinai entropy. Such dynamical quantifiers fluctuate around some positive value, till a sudden decrease occurs at the attraction to the regular attractor. These observations are consistent with the type-II supertransients often observed in spatially extended systems [10]. In a strong coupling regime, we have found traveling wave states up to the maximal nonlinearity $a = 2$. Thus the fully developed spatiotemporal chaos in this regime may belong to supertransients [14].

We note that the linear relationship between the asymmetry s and the velocity v (in section 2) is seen only for $a > a_{tr}$. This relationship may be partially explained from the results in the present section, although further studies are necessary for a complete explanation: For $a < a_{tr}$, the pattern selection can occur locally, and some local distortion in the wave pattern may not be removed. Then spatial asymmetry can remain even for a non-traveling attractor, and the s - v relationship can be very complicated. For $a > a_{tr}$, on the other hand, slight distortion in wave pattern leads to global chaotic transients. Only patterns without distortion are admissible as attractors. The s - v relationship may be expected to be monotonic and simpler.

—Fig. 19 —

—Fig. 20 —

³If a is not so large (near $a \approx a_{tr}$), we have often observed some local traveling wave patterns during the transients. This dynamics can be attributed to the spatiotemporal intermittency of type-II [9].

8 Switching among attractors with different velocities

By a suitable input at a site at one time step, we can make an external switch from one attractor to another (with a different velocity). By a local input, the structure of an attractor is changed over the whole lattice. Local information by an input is transformed into a *global* wave pattern (Fig.21). In the medium nonlinearity regime ($a < a_{tr}$), the switching process occurs within a short time, without any global chaotic transients.

As is expected this switching is easily attained by applying an input at site(s) in a phase slip. For example, assume that the phases at neighboring 5 domains are given by $[0, +\pi/2, +\pi, +3\pi/2, 2\pi]$. By applying an input at site(s) of the third domain, the phases can be switched to $[0, +\pi/2, 0, -\pi/2 (= 3\pi/2), 0]$. Thus a phase slip is removed, leading to a switch to an attractor with $v \rightarrow v_{next} = v - v_1$. We can control a switch by choosing an input site and value so that the number of phase slips is in(/de)creased.

For larger nonlinearity ($a > a_{tr}$), the chaotic transient lasts for many time steps during the course of switching. In this case the control of switching is almost impossible; it is hard to predict the length of the switching process or the attractor after the switch. This type of chaotic transients in the search for an attractor can be seen in some models with chaotic itinerancy [13] and in the neural activity in an olfactory bulb [15].

—Fig. 21 —

9 Fluctuating Domain by Chaos

In the weak coupling case, a frozen random pattern is observed [5] for weak nonlinearity ($a < 1.55$). In our strong coupling case, which phase corresponds to it? In this coupling regime, domains with variable sizes are again formed. These domains, however, are not fixed in space. The boundary of domains here fluctuates in time. Over some time steps some region moves in one direction locally, but then it changes the direction of traveling (see Fig.22a). In spatial return maps, the motion of $(x_n(i), x_n(i+1))$ along a curve changes its direction with time. The boundary motion is diffusive and Brownian-like (see Fig.22). Furthermore, the size of domains can also vary (chaotically). Domain distribution is rather random. We have plotted the spatial power spectrum $S(k) = \langle |\sum_j x_n(j) \exp(ikj)|^2 \rangle$ with the temporal average $\langle \dots \rangle$. In contrast with the peaks corresponding to the selected wavelength in the regular traveling wave regime, there are no clear peaks in the spectra (see Fig 24). The decay of the power spectra with the wavenumber is consistent with the diffusive motion of domains, while the decay in the frozen random phase in the weak coupling regime is much slower, due to the absence of such diffusive motion.

In this phase, some attractors have phase slips. Again a phase slip is defined by a unit with a sequence of domains of 2π phase advance. In an attractor with phase slips, the pattern moves (in average) in one direction with some fluctuation. Generally the pattern has some average velocity depending on the number of phase slips, although a fluctuating boundary of domains brings about the fluctuation of the velocity. Here chaos is not eliminated in large domains. In this case chaos is transported along with the traveling wave. Chaos localized in large domains moves together with the wave and in the same direction. An example of a pattern is given in Fig. 23 with the corresponding spatial return maps .

If a is smaller, domains of various sizes coexist, while the appearance of larger domains is less frequent as a approaches ≈ 1.55 , where pattern selection sets in.

Chaos in the internal dynamics in a large domain is confirmed by the Lyapunov spectra given in Fig. 25. The slope of the spectra is small at $\lambda \approx 0$. These exponents near 0 are

thought to come from the diffusive motion of the domains. Co-moving mutual information decays exponentially in space and time (Fig.26), implying that there are no remaining patterns in space and time.

—Fig. 22 —
—Fig. 23 —
—Fig. 24 —
—Fig.25 —
—Fig. 26 —

10 Propagating Kinks in Period-doubling Media

To clarify the difference between the phase slips and conventional solitons, we have studied our CML in the period-doubling regime with a strong coupling. As is known our CML exhibits the period-doubling of kink patterns [2]. In the lower coupling regime ($\epsilon < .4$), these kinks are pinned at their positions. In the strong coupling regime, some kinks can move when they form a phase gradient in one direction (see for example fig.27). This phase in-(de-)crease is possible if the period of the kinks is larger than 2 [18]. If the period is 4, for example, there can be a series of domains with the increase of the phase $0, \pi/2, 3\pi/2, 2\pi$, separated by 3 kinks. These kink patterns form a phase gradient, which drives them to move with a constant speed.

As for this phase advance, these moving kinks are similar to our phase slips. However, the kinks here are completely local. When there are two kinks at a distant position, they move almost independently with their original speeds, (until they collide). See Fig.27, where elimination of one kink by external input does not cause any change to the propagation of the other kink. Furthermore, there is no discrete selection of speeds. The speed of a kink gradually varies with the phase gradient within the kink pattern. In Fig. 27, change of a tail length at a kink leads to a slight change of speed. Kinks here belong to the same class as those studied in some partial differential equations like a ϕ^4 system. In an oscillatory medium (without chaos), we can expect the existence of kinks with period-doubling as in the present example.

—Fig. 27 —

11 Summary and discussions

In the present paper we have reported a traveling wave triggered by phase slips. The velocity of traveling attractors forms quantized bands determined by the number of phase slips. Frozen attractors (without any phase slips) and traveling attractors with different velocities coexist. The velocity of each band increases linearly with the number of slips. When the nonlinearity is large, the proportionality between the asymmetry of a pattern and the velocity holds even within each band. In this case, (approximate) symmetry is self-organized for frozen attractors.

Through pattern selection of domains with some wavenumbers, chaos is completely eliminated leading to quasiperiodicity. When there is a mismatch between the size and the wavelength, remaining frustration leads to a chaotic motion of the wave. If the frustration is large, chaotic itinerancy over many traveling (and frozen) states is observed. Our system itinerates over states with different velocities of traveling. The residence time in each state obeys a power law distribution. If the frustration is not so large, a chaotic traveling wave is observed, where the chaotic modulation is transmitted in the opposite direction to traveling wave

It should be noted that a local phase slip affects globally the motion of the total system. This is in strong contrast with the kink type propagation (also observed in our system in a non-chaotic region), where it propagates as a local quantity. The additivity of velocity of the wave with the number of phase slips is a clear manifestation of the global nature.

By local external inputs, one can create or destroy a phase slip, and to switch to an attractor with a different velocity. By the traveling wave, information is transmitted to the whole space within the steps in the order of $O(N)$. Thus the transformation from local to global information is possible through this switching, which may be useful for information processing and control.

We have noted two types of transient processes in the course of attraction to traveling states. When the parameter for the nonlinearity is large, a supertransient (with quasistationary measure) is observed whose transient length increases exponentially with the system size, while such rapid increase is not seen in the medium nonlinearity regime ($a < 1.75$).

In the weaker nonlinearity regime ($a < 1.55$) corresponding to the frozen random pattern, we have found fluctuation of domain sizes and Brownian-like motion of domains. Coexistence of fluctuating domains and phase slips is also noted. In this random pattern, chaos is localized in large domains, and propagates along the traveling wave.

We have analyzed the dynamics of the traveling wave with the use of spatial return maps, Lyapunov spectra, and co-moving mutual information flow. When chaos is suppressed in the pattern selection regime, the maximal Lyapunov exponent is zero implying that the traveling wave attractor is on a torus whose dimension depends on the number of phase slips. This null Lyapunov exponent remains even in a chaotic wave or chaotic itinerancy. This exponent is due to the Goldstone-type mode corresponding to the traveling structure. Through the mutual information flow, we note the creation and transmission of information by a chaotic traveling wave. The chaotic modulation added on the traveling wave leads to the possibility of information transmission, created by chaos [16].

There is no apriori reason to deny the possibility of the traveling wave in partial differential equation systems. The existence of admissible velocity bands is thought to be due to the suppression of chaos. For convenience of an illustration, consider a partial differential equation with two components

$$\partial \vec{\phi}(r, t) / \partial t = \vec{F}(\vec{\phi}(r, t)) + D \nabla^2 \vec{\phi}(r, t). \quad (4)$$

If there exists a traveling wave solution $\vec{\phi}(r, t) = \vec{f}(r - vt)$, it must satisfy

$$-v \vec{f}' = \vec{F}(\vec{f}) + D \vec{f}'' . \quad (5)$$

If this coupled second order differential equation has a periodic solution for a range of velocities v , then the traveling wave $\vec{f}(r - vt)$ can be a solution of eq.(4). Generally speaking, nonlinear eq.(5) has a chaotic solution for some range of v , and has windows of limit cycles in the parameter space (v) for chaos. This scenario implies the existence of admissible velocity bands for stable traveling wave solutions, as in our CML example.

Traveling waves have often been studied in various experiments [19]. Quite recently, Croquette's group has observed traveling waves in Bénard convection with periodic boundary condition. Indeed this traveling wave is triggered by a unit of a 2π phase advance [21]. They have also observed chaotic itinerancy over different traveling states, when the motion in a Bénard cell is strongly chaotic [22]. It is expected that this discovery belongs to the same class as our traveling wave. It is interesting to check if attractors with different velocities coexist by applying perturbations to such experimental systems. A search for our velocity bands in experiments will also be of interest. Detailed comparison with

In the weak nonlinearity regime, the suppression of chaos is not possible, where domains show chaotic Brownian motion without a traveling velocity. This type of floating domains has some correspondence with the dispersive chaos found in Bénard convection by Kolodner's group [20].

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References

- [1] Rapid communication of the present studies is given in K. Kaneko, Phys.Rev. Lett. 69 (1992) 905
- [2] K. Kaneko, Prog. Theor. Phys. 72 (1984) 480, 74 (1985) 1033
- [3] K. Kaneko, Ph. D. Thesis *Collapse of Tori and Genesis of Chaos in Dissipative Systems*, 1983 (enlarged version is published by World Sci. Pub., 1986)
- [4] K. Kaneko, Physica 23D (1986) 436
- [5] K. Kaneko, Physica 34D (1989) 1; 36D (1989) 60
- [6] J. P. Crutchfield and K. Kaneko, "Phenomenology of Spatiotemporal Chaos", in *Directions in Chaos* (World Scientific, 1987)
- [7] K. Kaneko "Simulating Physics with Coupled Map Lattices — Pattern Dynamics, Information Flow, and Thermodynamics of Spatiotemporal Chaos" in *Formation, Dynamics, and Statistics of Patterns* (ed. K. Kawasaki, A. Onuki, and M. Suzuki, World. Sci. 1990)
- [8] I. Waller and R. Kapral, Phys. Rev. 30A (1984) 2047; R. Kapral, Phys. Rev. 31A (1985) 3868
- [9] for type-I spatiotemporal intermittency, see [1],[6], and H. Chaté and P. Manneville, Physica 32D (1988) 409.
for type-II intermittency, see J. D. Keeler and J. D. Farmer, Physica 23D (1986) 413; [5].
See also P. Grassberger and T. Schreiber, Physica 50 D (1991) 177
- [10] J.P. Crutchfield and K. Kaneko, Phys. Rev. Lett. 60 (1988) 2715 and in preparation: K. Kaneko, Phys. Lett. 149A (1990) 105
- [11] K. Kaneko, Prog. Theor. Phys. Suppl. 99 (1989) 263
- [12] R. J. D. ... and K. Kaneko, Phys. Lett. 110A (1987) 207

- [13] K. Kaneko, Phys. Rev. Lett. 63 (1989) 219; Physica 41D (1990) 38; K. Ikeda, K. Matsumoto, and K. Ohtsuka, Prog. Theor. Phys. Suppl. 99 (1989) 295; I. Tsuda, in *Neurocomputers and Attention*, (eds. A.V. Holden and V. I. Kryukov, Manchester Univ. Press, 1990)
- [14] F. Willeboordse, Phys. Rev. E (Brief Reports), in press (1992); F. Willeboordse, PhD thesis, Tsukuba Univ., to be submitted
- [15] W. Freeman and C. A. Skarda, Brain Res. Rev. 10 (1985) 147; W. Freeman, Brain Res. Rev. 11 (1986) 259
- [16] R. Shaw, Z. Naturforschung, 36a (1981) 80; *The Dripping Faucet as a Model Chaotic System*, (Aerial Press, Santa Cruz, 1984)
- [17] K. Ikeda and K. Matsumoto, Phys Rev Lett. 62 (1989) 2265
- [18] This type of moving kink domains is first discussed in C. Bennett, G. Grinstein, Y. He, C. Jayaprakash and D. Mukamel, Phys. Rev. A 41 (1990) 1932. In the case of period-2 attractor, possible attributed phases are π or $-\pi$. Thus it is impossible to distinguish between the phase increase or decrease (for example the phase changes with the sequence of $\pi, -\pi, \pi$ can have neither phase advance or retardation.)
- [19] D.R. Ohlsen et al., Phys. Rev. Lett. 65 (1990) 1431; P. Kolodner, Phys. Rev. A 42 (1990) 2475 ; V. Croquette and H Williams Physica 37 D (1989) 295; D. Bensimon et al., J. Fluid Mech. 217 (1990) 441
- [20] P. Kolodner, J.A. Glazier, and H Williams, Phys. Rev. Lett. 65 (1990) 1579
- [21] V. Croquette, J.M Flesselles and S. Jucquois, private communication
- [22] Yanagita has recently observed the traveling wave in Bénard convection in the CML model for convection given by T. Yanagita and K. Kaneko, "CML model for convection", submitted to Phys. Lett. A.

Fig.1

Amplitude-space plot of $x_n(i)$ with a shift of time steps. 200 sequential patterns $x_n(i)$ are displayed with time (per 64 time steps), after discarding 25600 initial transients, starting from a random initial condition. $a = 1.71, \epsilon = .5$, and $N = 64$. Four examples of attractors. (a) $v_p = 0$ (b) $v_p = -v_1$ (c) $v_p = v_1$ (d) $v_p = v_2$

Fig.2 Asymmetry s versus velocity v_p for attractors started from randomly chosen 300 initial conditions. The asymmetry and velocity are computed from the average of 160000 steps after discarding 100000 initial transients. $N = 64$ and $\epsilon = .5$. (a) $a = 1.64$ (b) $a = 1.69$ (c) $a = 1.75$ (d) $a = 1.8$

Fig.3 The basin volume for each attractor with the velocity v_p , calculated from 2000 random samples, for $a = 1.72, \epsilon = .5$, and $N = 100$. Obtained from 200000 steps after discarding 200000 steps. The number of initial conditions fallen on the attractor with the velocity v_p is plotted as a function of v_p . (b) The basin volume ratio for attractors in each velocity band calculated from the data for (a).

Fig.4 Velocities of attractors versus asymmetry s , obtained with the algorithm in the text, applied per 32 steps, over 32x5000 steps, after discarding 50000 initial transients. Velocities from randomly chosen 500 initial conditions are overlayed. $N = 100$. Data for $a = 1.66, 1.67 \dots, 1.85$, are overlayed, for $\epsilon = .5$. (additional data are included from fig.5 in [1]).

Fig.5 The absolute values of velocities $|v_p|$ of attractors, plotted as a function of size N . The velocities are computed with the algorithm in the text, applied per 32 steps, over 32x5000 steps, after discarding 50000 initial transients. Velocities from randomly chosen 50 initial conditions are overlayed. $a = 1.73$, and $\epsilon = .5$.

Fig.6 Basin ratio for traveling wave as a function of ϵ , for $a = 1.69$. Velocity of attractors from randomly chosen 50 initial conditions are examined, to count the number of attractors with $v \neq 0$.

Fig.7 Space-Amplitude plot of $x_n(i)$. $a = 1.72, \epsilon = .5$, and $N = 64$. 100 steps are overlayed after discarding 10000 initial transients. This wave pattern is traveling to the left direction. (b) The same plot, shown per 4 steps. Arrows indicate the phase of oscillation of the corresponding domains.

Fig.8: Spatial Return Map: $\{x_n(1), x_n(2)\}$ are plotted over the time steps $n = 10001, 10002, \dots, 210000$. $a = 1.70, \epsilon = .5$, and $N = 64$.

Fig.9: Lyapunov spectra of our model with $\epsilon = .5$, starting with random initial conditions, discarding 50000 initial transients. The calculation is carried out through the products of Jacobi matrices over 16384 time steps. $N=64$. $a = 1.72$; for attractors with 2 phase slips (solid line) and one phase slip (two examples; dotted lines), and frozen attractors without a phase slip (two examples: broken lines)

Fig.10: Amplitude-space plot of $x_n(i)$ with a shift of time steps. 200 sequential patterns $x_n(i)$ are displayed with time (per 128 time steps), after discarding 10240 initial transients, starting from a random initial condition. $a = 1.69, \epsilon = .5$, and $N = 92$.

Fig.11: Spatial Return Map: $\{x_n(1), x_n(2)\}$ are plotted over the time steps $n =$

10001, 10002, \dots 210000. $a = 1.69$, $\epsilon = .5$, and $N = 100$.

Fig.12: Lyapunov spectra of our model with $\epsilon = .5$, starting with a random initial condition, discarding 50000 initial transients. The calculation is carried out through the products of Jacobi matrices over 32768 time steps. $N=100$. $a=1.69$: for attractors with $v = v_1$ (solid or dotted line) and $v = 0$ (broken line).

Fig.13: Amplitude-space plot of $x_n(i)$ with a shift of time steps. 200 sequential patterns $x_n(i)$ are displayed with time (per 1024 time steps), after discarding 1024000 initial transients, starting from a random initial condition. $a = 1.69$, $\epsilon = .5$, and $N = 51$.

Fig.14: Spatial Return Map: $\{x_n(1), x_n(2)\}$ are plotted over the time steps $n = 10001, 10002, \dots 210000$. $a = 1.69$, $\epsilon = .5$, and $N = 51$.

Fig.15 Residence time distribution for a state with $v \approx v_k$ in the chaotic itinerancy of traveling wave. The distribution is taken over 819200 time steps after 20000 initial transients, and sampled over 500 initial conditions. $a = 1.69$, $\epsilon = .5$, and $N = 51$. (a) $k = 0$ (staying at a frozen state) (b) $k = 1$ (residence at a one-phase-slip-state) (c) $k = -1$ (residence at a one-negative-phase-slip-state) (d) $k = 2$ (residence at a two-phase-slip-state)

Fig.16: Lyapunov spectra of our model with $\epsilon = .5$, starting with a random initial condition, discarding 50000 initial transients. The calculation is carried out through the products of Jacobi matrices over 32768 time steps. $N=51$. $a=1.69$: Spectra from three different initial conditions are overlayed.

Fig. 17: Co-moving mutual information flow for the logistic lattice (1), obtained with the algorithm in the text. $I(m, t_c \times j)$ is plotted for $-10 \leq m \leq 10$ and $0 \leq j \leq 15$ with coarsegraining time $t_c = 256$. The probability is calculated using 64 bins and sampled over $5000xt_c$ steps over the whole lattice. (a) $a = 1.69$, $\epsilon = .5$ and $N = 100$: for a chaotic attractor with $v_p = v_1$. (b) $a = 1.69$, $\epsilon = .5$ and $N = 100$: for an attractor with $v_p = -3v_1$. (c) $a = 1.69$, $\epsilon = .5$ and $N = 100$: for an attractor with $v_p = 0$.

Fig. 18: Co-moving mutual information flow for the logistic lattice (1), obtained with the algorithm in the text. $I(m, t_c \times j)$ is plotted for $-10 \leq m \leq 10$ and $0 \leq j \leq 16$ with coarsegraining time $t_c = 256$. The probability is calculated using 64 bins and sampled over $5000xt_c$ steps over the whole lattice. $a = 1.69$, $\epsilon = .5$ and $N = 51$

Fig.19: Space-time diagram for the coupled logistic lattice (1), with $\epsilon = 0.5$, and starting with a random initial condition. If $x_n(i)$ is larger than x^* (unstable fixed point of the logistic map), the corresponding space-time pixel is painted as black (if $x > x_2^* \equiv (\sqrt{1+7a}-1)/(2a)$ painted darker), while it is left blank otherwise.

(a) $a = 1.71$, $N = 300$. Every 128th step is plotted. (b) $a = 1.73$, $N = 300$. Every 256th step is plotted. (c) $a = 1.76$, $N = 200$. Every 2048th step is plotted.

Fig.20: Lyapunov spectra of our model with $\epsilon = .5$, starting with a random initial condition: Comparison of quasistationary states with an attractor. In the former, two sets of spectra in the transients states are calculated after discarding 10000 initial transients, for two different initial conditions. They are overlayed, but agree within the linewidth of the figure. For the latter, the data after 1000000 steps are adopted. The calculation is carried out through the products of Jacobi matrices over 32768 time steps. $N = 50$, $a =$

Fig.21: Switching process; Amplitude-space plot of $x_n(i)$ with a shift of time steps. 200 sequential patterns $x_n(i)$ are displayed with time (per 128 time steps), after discarding 40960 initial transients, starting from a random initial condition. At the time steps and lattice point indicated by the arrows, external input is applied to change the value of $x_n(i)$ at the corresponding i and n . $\epsilon = .5$, and $N = 64$ (a) $a = 1.7$ (b) $a = 1.75$.

Fig.22: Amplitude-space plot of $x_n(i)$ with a shift of time steps. 200 sequential patterns $x_n(i)$ are plotted with time (per 1024 time steps), after discarding 20480 initial transients, starting from a random initial condition. $\epsilon = .5$, and $N = 100$. (a) $a = 1.47$ (b) $a = 1.52$.

Fig. 23: Spatial return maps with the corresponding space amplitude plots: $(x_n(1), x_n(2))$, $(x_n(13), x_n(14))$, $(x_n(25), x_n(26))$, and $(x_n(37), x_n(38))$ are plotted over the time steps $n = 12800, 12801, \dots, 64000$, while $x_n(i)$ is plotted with time per 256 steps. $a = 1.5, \epsilon = .5$, and $N = 50$; (a)without any phase slip. (b)with one phase slip.

Fig. 24: Spatial power spectra $S(k)$ obtained from the Foureir transform of pattern $x_n(i)$. Calculated from the average over 100000 time streps after discarding initial 10000 steps. $\epsilon = .5$, and $N = 2048$; (a) $a = 1.5$ (b) $a = 1.65$.

Fig.25: Lyapunov spectra of our model with $\epsilon = .5$, starting with a random initial condition. Three examples of calculation are overlayed starting from 3 randomly chosen initial conditions, after discarding 10000 initial transients. Calculated over 32768 time steps. $N = 50$. $a = 1.53$.

Fig. 26: Co-moving mutual information flow for the logistic lattice (1), obtained with the algorithm in the text. $I(m, t_c \times j)$ is plotted for $-10 \leq m \leq 10$ and $0 \leq j \leq 16$ with coarsegraining time $t_c = 256$. The probability is calculated using 64 bins and sampled over $5000xt_c$ steps over the whole lattice. $a = 1.5, \epsilon = .5$ and $N = 100$

Fig. 27 Amplitude-space plot of $x_n(i)$ (for a moving kink) with a shift of time steps. 200 sequential patterns $x_n(i)$ are depicted with time (per 1024 time steps), after discarding 4096 initial transients, starting from a random initial condition. At the time steps and lattice points indicated by the arrows, the value of $x_n(i)$ at the corresponding i and n is shifted to 0 by an input. $a = 1.4, \epsilon = .5$, and $N = 64$.